

# Statistics of heat generated in a solvable dissipative Landau-Zener model.

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We consider an adiabatic Landau-Zener model of two-level system diagonally coupled to an Ohmic bosonic bath of large spectral width and derive through fermionization its exact solution at a special value of the coupling constant. From this solution we obtain the characteristic function of the distribution of energy transferred to the bath during the evolution of the system ground state as a functional determinant of a single particle operator. At zero temperature this distribution is further found to be exponential and at finite temperature the first three moments of the distribution are calculated.

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Recently the interest to dissipative Landau-Zener (LZ) models has been revived in the context of studying fluctuation relations [1–4] for work and dissipation in small systems driven out of equilibrium by external force. The main focus of the renewed interest is, however, different from the earlier studied [5, 6] effect of a dissipative environment on the LZ probability of the non-adiabatic transition. Now, it is the quantum statistics of the energy transferred to the environment. This interest has grown in connection with the definition and measurement of the work performed on a small quantum system in the non-equilibrium process in the quantum version [7–9] of the fluctuation relations. It has been suggested [10] based on the principle of conservation of energy to define this work through the heat generated in the environment. To study this matter a simple, but experimentally feasible system of superconducting Cooper-pair box driven by a gate voltage has been considered [11, 12] in the regime when its theoretical description reduces to a dissipative LZ model of two levels undergoing avoiding crossing and coupled to an Ohmic bosonic bath. In spite of its simplicity a non-perturbative quantum solution to this interacting model still remains a challenging task, in particular, since at low energy the model is equivalent [13] to an anisotropic Kondo model driven out of the thermal equilibrium by the time dependent magnetic field or, equally, to an interacting resonant level model (IRLM) with the time dependent level energy [14]. Their stationary equilibrium solution, in general, is available only in the Bethe ansatz technique [15]. Its generalization to the stationary non-equilibrium IRLM of electronic transport is difficult [16, 17] and remains completely unknown to the non-stationary models.

In this work therefore we consider a special case of this dissipative LZ model at a particular value of the bath coupling constant which corresponds to the Toulouse limit of the anisotropic Kondo model solvable through re-fermionization. In equilibrium this special case of the Kondo model has been particular important since it gives a simple but universal description of the low energy Fermi liquid behavior characterizing an antiferromagnetic fixed point for the renormalization group scaling procedure [18]. Therefore this particular LZ model of the two-level

system will also show the general low energy properties of the heat distribution generated during the system evolution. We will construct solution to this model and use it to calculate the heat distribution in the adiabatic limit when the system enters and exits the evolution in its ground states. In this limit the excitations produced in the Ohmic environment of wide energy spectrum are limited to smaller energies than the environment spectral width.

*Model* - The LZ model (also known [19] as the Landau-Zener-Stückelberg-Majorana model) describes transition of the system between its two states denoted as spin up (down)  $|\uparrow(\downarrow)\rangle$  with the time dependent Hamiltonian  $\mathcal{H}_S(t) = at\sigma_z/2 + \Delta\sigma_x$ . Here  $\sigma_{x(z)}$  are Pauli matrices and the constant sweep velocity  $a(>0)$  regulates crossing of the diabatic energies  $\pm at/2$  of the two states coupled by the tunneling amplitude  $\Delta$ . The interaction of the system with the environment modeled as a bath of the harmonic oscillators is introduced by the additional part of the Hamiltonian ( $\hbar = 1$ )

$$\mathcal{H}_E = \int \frac{dx}{4\pi} (\partial_x \phi)^2 + U \sigma_z \partial_x \phi(0), \quad (1)$$

where annihilation and creation operators of the oscillators are combined into the bosonic chiral field  $\phi(x) = \int d\omega \exp(-i\omega x) \phi(\omega)$ . Its Fourier components satisfy  $[\phi(-\omega'), \phi(\omega)] = \delta(\omega - \omega')/\omega$ . Therefore the spectral function of the bath is defined [5] by the correlator of the coupling operator in Eq. (1) as  $J(\omega) = (2U)^2 \omega \exp(-\omega/D)$ , where the energy cut-off  $D$  is assumed to be much larger than all other energy parameters in the model, in particular,  $D \gg \Delta$ . This smooth exponential cut-off of  $J(\omega)$  substitutes for a more realistic Lorentzian one used in Ref. [11].

By applying the unitary transformation  $\mathcal{U} = \exp(i\phi(0)\sigma_z/2)$  to the sum of both parts of the Hamiltonian  $\mathcal{U}^\dagger[\mathcal{H}_S + \mathcal{H}_E]\mathcal{U}$  and making use of the fermionic representation of Pauli matrices  $\sigma_z = 2d^\dagger d - 1$ ,  $\sigma_x = \sigma_+ + \sigma_- = d^\dagger \eta + \eta d$ , where  $\eta$  denotes an auxiliary Majorana fermion and  $d$  is the annihilation operator of another fermion we come to the fermionic description of the

model by the time-dependent IRLM Hamiltonian

$$\begin{aligned}\mathcal{H}_F(t) &= \mathcal{H}_0 + at(d^+d - \frac{1}{2}) + w(d^+\psi(0) + h.c.) \\ &\quad + \pi(2U - 1)\psi^+(0)\psi(0)(2d^+d - 1), \quad (2) \\ \mathcal{H}_0 &= -i \int dx \psi^+(x) \partial_x \psi(x),\end{aligned}$$

where the chiral Fermi field  $\psi(x)$  stands for  $\psi(x) = \sqrt{\frac{D}{2\pi}} \eta e^{i\phi(x)}$ , the Fermi sea of occupied fermion states is defined by the zero chemical potential, and the tunneling amplitude  $w$  is  $w = \Delta \sqrt{2\pi/D}$ . The density of the Fermi states also undergoes the same exponential cut-off at large absolute values of their energies as the bosonic bath modes. In this formulation of the problem the case of the system being at  $t = -\infty$  in the ground state, which we will study below, corresponds to the filled resonant level entering the Fermi sea from its underneath. The exponential boundaries  $\pm D$  of the Fermi states energies do not appear in the Hamiltonian (2) directly and in this form it describes only evolution of the states with energies deep inside the energy band. In the stationary case  $a = 0$  this description is accurate since the tunneling rate  $\gamma = w^2/2$  is small  $\gamma \ll D$ . Although in our time dependent IRLM the resonant level traverses the whole fermion energy band the use of this Hamiltonian is still justified if the tunneling in and out of the resonant level vanishes quickly enough far from the Fermi level and the energies of the Fermi sea excitations remain much less than  $D$ .

*Fermion model solution* - We will find solution to the model described by Eq. (2) at  $U = 1/2$ , when the contact interaction between the resonant level and the Fermi sea vanishes. Then the equations of motion for the fermion operators can be written as follows:

$$\begin{aligned}i\partial d(t) &= atd(t) + \frac{w}{2}(\psi_{in}(t) + \psi_{out}(t)), \quad (3) \\ \psi_{out}(t) &= \psi_{in}(t) - iwd(t),\end{aligned}$$

where the incoming and outgoing fermions  $\psi_{in(out)}$  describe the chiral propagation of  $\psi$  on both sides from the resonant level as  $\psi(x, t) = \theta(-x)\psi_{in}(t-x) + \theta(x)\psi_{out}(t-x)$ . Further solving the linear differential equation we express the outgoing fermion operators at time  $t$  through the incoming operators at earlier times starting from the initial one  $t_0$

$$\begin{aligned}\psi_{out}(t) &= \psi_{in}(t) - 2\gamma e^{-iE(t)} \int_{t_0}^t d\tau e^{\gamma(\tau-t) + iE(\tau)} \psi_{in}(\tau) \\ &\quad - iwe^{\gamma(t_0-t)} e^{i(E(t_0)-E(t))} d(t_0), \quad E(t) = at^2/2 \quad (4)\end{aligned}$$

Here  $t_0$  can be chosen as early as the entrance time of the level into the fermion band:  $t_0 \approx -D/a$ . Then under assumption that the traversal time  $D/a$  is much larger than the tunneling time and  $\gamma D/a \gg 1$ , decay of the resonant level state makes contribution from the last term in Eq. (4) negligibly small at finite time and the low limit of

integration in the second term may be drawn to  $-\infty$ . In the resultant relation between the incoming and outgoing fermions it is convenient to represent both Fermi fields as  $\psi_\alpha(t) = \int dk \exp(-i[kt - k^2/(2a)]) c_\alpha(k)/\sqrt{2\pi}$ ,  $\alpha = in, out$ . In this representation the  $S$  matrix relating incoming and outgoing plane waves in this scattering problem  $c_{out}(k) = \int dk' S(k, k') c_{in}(k')$  follows from Eq. (4) in the simple form:

$$S(k, k') = \delta(k - k') - \frac{2\gamma}{a} \theta(k - k') e^{\gamma(k'-k)/a}. \quad (5)$$

This  $S$  matrix is unitary in the absence of the energy band restrictions on the fermion energies  $k$ . Therefore it properly describes the low energy scattering under the same assumption of the large enough traversal time when we can neglect the exponentially small probability  $\exp(-2\gamma D/a)$  for a low energy fermion to reach the energy band boundary. Since  $\gamma D/a = \pi \Delta^2/a$  this is also the LZ probability of the non-adiabatic transition, which at zero temperature is not affected [5, 6, 20] by the diagonal coupling of the system to the bath in Eq. (1).

*Characteristic function of heat* - Under this adiabatic assumption we can limit our consideration only to the evolution of the Fermi sea. Then the characteristic function of the distribution of its energy excitations  $\chi(\lambda)$  can be expressed through the  $S$  matrix following the lines of derivation [21] of the Levitov-Lesovik formula [22] for the full counting statistics of charge transfer. The result comes as a determinant of the operator acting in the one-particle Hilbert space:

$$\chi(\lambda) = \det\{1 + n_F(e^{-i\lambda h_0} S^+ e^{i\lambda h_0} S - 1)\}, \quad (6)$$

where  $h_0$  stands for the one-particle Hamiltonian operator  $h_0(k, k') = k\delta(k - k')$  and the operator  $n_F$  is defined by the correspondent Fermi-Dirac distribution function. As follows from its derivation and properties the  $S$  matrix (5) implies the constant density of the fermion states and hence the infinite depth of the Fermi sea. However, the functional determinant is well defined only for the operator whose difference from the identity is a trace class operator. Although we give below its general proper regularization the direct use of the expression Eq.(6) is also convenient and possible, if we impose restriction on occupation of the fermion states below some energy  $-W$  through introduction of an artificial filling factor  $\rho(k) = \exp(-|k|/W)$ , which will be lifted at the end of the calculations as  $W \rightarrow -\infty$ . In this way all moments of the heat distribution can be found by taking derivatives of the function

$$\ln \chi(\lambda) = \text{tr}\{\ln[1 + n(e^{-i\lambda h_0} S^+ e^{i\lambda h_0} S - 1)]\} \quad (7)$$

with respect to  $i\lambda$  at  $\lambda = 0$ , where  $n = \rho n_F$  is the initial one-particle density operator diagonal in the energy representation and  $\text{tr}$  assumes the uniform summation over all  $k$  energy states. Then the first derivative gives the average of the heat generated in the environment as

$$< Q > = \text{tr}\{n(S^+ h_0 S - h_0)\} = \text{tr}\{(S n S^+ - n) h_0\}. \quad (8)$$

Both expressions for the average heat in (8) are equivalent for the trace convergent density operators, though the second one permits lifting the filling factor restriction because it distinguishes contributions from the low and high energy excitations of the Fermi sea. Indeed, the variation of the one-particle density operator  $\Delta n \equiv S n S^+ - n$  is equal to

$$\Delta n(k, k') = \frac{2\gamma}{a} e^{-\gamma|k-k'|/a} \Delta n_{||}(\min\{k, k'\}) , \quad (9)$$

$$\Delta n_{||}(p) = -n(p) + \frac{2\gamma}{a} \int_{-\infty}^p dp' n(p') e^{2\gamma(p'-p)/a} .$$

At zero temperature the substitution of  $n = \rho n_F$  in Eq. (9) leads to the following expression

$$\Delta n = -\frac{2\gamma\theta(-k_{<})}{2\gamma W + a} e^{-\frac{\gamma}{a}|k-k'| + \frac{k_{<}}{W}} + \frac{4\gamma^2}{a} \frac{W\theta(k_{<})}{2\gamma W + a} e^{-\frac{\gamma}{a}(k+k')} , \quad (10)$$

where  $k_{<} = \min\{k, k'\}$ . It shows that any large energy cut-off  $W$  of the filling factor still produces some variations of the density deep inside the Fermi sea, which compensate its variations at small positive energies to insure the particle conservation:  $tr\Delta n = 0$ . Indeed, the level rising into the Fermi sea with the unfilled states below it is empty. On the other hand, by drawing first the cut-off  $W$  to the infinity in Eq. (10) we eliminate all density variations at negative energies and the density variation operator becomes

$$\Delta n_F(k, k') = \frac{2\gamma}{a} \theta(k)\theta(k') e^{-\gamma(k+k')/a} , \quad (11)$$

but with  $tr\Delta n_F = 1$ . This density variation operator describes evolution of the Fermi sea caused by the filled level rising into it and bringing an additional fermion when the system is in the ground state at  $t = -\infty$ . In this case we find from Eq. (11) that under the adiabatical assumption all excitations produced in the Fermi sea have energies much less than  $D$ , which is consistent with our use of the S-matrix (5).

At finite temperature and  $n = n_F$  in Eq. (9) the function  $\Delta n_{||}$  defining the diagonal matrix elements of the density variation operator can be found through Laplace transformation as

$$\Delta n_{||}(k) = \frac{1}{2\pi i} \int_C \frac{ds e^{-sk} \pi T s}{\sin(\pi T s) (\frac{2\gamma}{a} - s)} , \quad (12)$$

where the contour  $C$  coincides with the imaginary axis infinitely shifted to the right. We confirm from Eqs. (9,12) that  $tr\Delta n_F = 1$  does not depend on temperature and so does the average heat  $\langle Q \rangle = a/(2\gamma) = Da/(2\pi D^2)$  in Eq. (8).

Generalizing this method we would obtain the whole distribution of the heat produced during the evolution of the system ground state if we managed to transform the general expression (6) for the characteristic function into the form, which permits lifting the filling factor restriction. This can be done by using a regularization procedure similar to that developed in Refs. [23]

and [24] for calculation of the charge transfer statistics in transport problems. To implement it we multiply the determinant in Eq. (6) from the left and from the right by the mutually canceling factors  $\det \exp\{i\lambda n_F h_0\}$  and  $\det \exp\{-i\lambda(n_F h_0)_S\}$ , respectively, where we denote  $S^+(n_F h_0)S \equiv (n_F h_0)_S$ . The result can be brought into the form:

$$\chi(\lambda) = \det\{e^{i\lambda n_F h_0} (1 - n_F) e^{-i\lambda(n_F h_0)_S} + e^{-i\lambda(1-n_F)h_0} n_F (e^{i\lambda((1-n_F)h_0)_S})\} , \quad (13)$$

which remains well defined for the infinitely deep Fermi sea without any additional restrictions. We further demonstrate this with the zero temperature calculations.

*Zero temperature heat distribution* - Since at zero temperature the density operator  $n_F$  becomes a projector operator, the characteristic function in Eq. (13) transforms into the following one:

$$\chi(\lambda) = \det\{1 + S(1 - n_F)S^+ n_F (e^{-i\lambda h_0} - 1) + S n_F S^+ (1 - n_F) (e^{i\lambda h_0} - 1)\} . \quad (14)$$

Further substitution here  $S n_F S^+ = n_F + \Delta n_F$  with  $\Delta n_F$  from Eq. (11) makes it possible to calculate logarithm of the determinant in Eq. (14) as follows:

$$\ln \chi(\lambda) = \ln(1 + \frac{ia\lambda}{2\gamma - ia\lambda}) . \quad (15)$$

Its  $l$ 'th derivative with respect to  $i\lambda$  at  $\lambda = 0$  or the  $l$ 'th order semi-invariant defines the correspondent reduced correlator  $\langle\langle Q^l \rangle\rangle$  of the energy dissipated during the ground state evolution of the two-level system as  $\langle\langle Q^l \rangle\rangle = (l-1)!(a/2\gamma)^l$ .

Fourier transformation of the characteristic function in Eq. (15) gives the exponential distribution for the dissipated energy

$$P(Q) = \theta(Q) \frac{2\gamma}{a} e^{-2\gamma Q/a} . \quad (16)$$

This distribution coincides with the heat distribution derived [11] from solution of the master equation describing the same model. The derivation is based on calculation of the probability to find the resonant level positioned at the energy  $Q$  to be occupied. Therefore the coincidence is expected at large dissipated energies but not at small  $Q$ , where the master equation solution for the probability becomes incorrect. Notice also that the distribution in Eq. (16) assumes the strictly adiabatic transition because of its normalization.

*Second and third semi-invariants of the heat distributions* - We will use Eq. (7) to find the second and third derivatives of  $\ln \chi$  at  $\lambda = 0$ . The second derivative comes as follows:

$$(-i\partial_\lambda)^2 \ln \chi = tr\{n\Delta h(1-n)\Delta h\} . \quad (17)$$

Here  $\Delta h = S^+ h_0 S - h_0$  is the operator of the one-particle energy variation. Substituting the  $S$  matrix from Eq. (5) we find its kernel to be equal to

$$\Delta h(k, k') = e^{-\gamma|k-k'|/a} . \quad (18)$$

Making use of it we transform the right side of Eq. (17) into a double integral over the energies, where the filling factor cut-off  $W$  can be drawn to the infinity. After we put  $n = n_F$  in Eq. (17) only the particle states close to the Fermi level contribute to the integral. The integration gives us the second reduced correlator of the heat as

$$\langle\langle Q^2 \rangle\rangle = \left(\frac{a}{2\gamma}\right)^2 + 2T^2\psi'(1 + \frac{2\gamma T}{a}), \quad (19)$$

where  $\psi'$  is the first derivative of the di-gamma function. From the known asymptotics of this function it follows that  $\langle\langle Q^2 \rangle\rangle = (a/2\gamma)^2 + 2\zeta(2)T^2$  at small temperature ( $\zeta(x)$  is the zeta function) and  $\langle\langle Q^2 \rangle\rangle = (Ta/\gamma)(1 + O(a^2/(2\gamma T)^2))$  at large one.

The result of our calculation of the third derivative can be first brought into the following form:

$$(-i\partial_\lambda)^3 \ln \chi = \text{tr}\{n\Delta h(1-n)\Delta h(1-n)\Delta h\} - \text{tr}\{n\Delta h n \Delta h(1-n)\Delta h\} + \text{tr}\{n\Delta h[h_0, \Delta h]\}. \quad (20)$$

Here the first two terms remain finite after substitution of the density matrix  $n = n_F$  and cancel each other because of the particle-hole symmetry. The third term however is ill-defined. To regularize it consistently with Eq. (13) we substitute the identity operator in the form  $1 = n + (1 - n)$  between the first  $\Delta h$  operator and the commutator. Since  $\text{tr}\{n\Delta h n[h_0, \Delta h]\} = 0$  we find the third reduced correlator of the heat to be equal to:

$$\langle\langle Q^3 \rangle\rangle = \int dk n_F(k) \int dk' (1 - n_F(k')) e^{-2\frac{\gamma}{a}|k-k'|} \times (k' - k) = \frac{1}{4} \left(\frac{a}{\gamma}\right)^3. \quad (21)$$

It does not depend on the temperature and coincides with the zero temperature expression found above.

In conclusion, we have considered the adiabatic LZ model of evolution of the two-level system diagonally coupled to an Ohmic bosonic bath of large spectral width  $D$  and derived through fermionization its exact solution at the coupling constant  $U = 1/2$ . From this solution for the system starting evolution in the ground state and generating only bosonic excitations of the energies much less than  $D$  we have obtained the characteristic function of the distribution of heat energy  $Q$  transferred to the bath as a functional determinant of a single particle operator. The determinant has been used to find that the distribution is exponential at zero temperature and to calculate its first three moments at finite temperature. The Fermi liquid behavior of this particular model is common for the LZ model at arbitrary  $U$ , which means that at low energy the heat distribution is an integer function of  $Q$  with the leading linear decrease and has a  $T^2$  temperature growth of its dispersion.

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